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AN ELASTODYNAMIC SOLUTION FOR AN ANISOTROPIC HOLLOW SPHERE

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Abstract—An exact solution for the problem of radial vibrations and dynamic stresses in an anisotropic hollow sphere acted on by a dynamic interior and exterior pressure is obtained. Formulas for spherically symmetric anisotropic problems are derived and results are carried out for some practical examples in which an anisotropic hollow sphere is subjected to a sudden load and an exponential decaying shock pressure. The features of the solution are related to the propagation of the spherical wave, and the radial vibrations of the anisotropic sphere are discussed.

1. INTRODUCTION

The study of radial vibrations and dynamic stresses in an elastic hollow sphere subjected to exterior and interior time-dependent pressures is a typical elastodynamic problem, and is very useful in engineering applications. Analyses and calculations for an isotropic spherical shell under a radially dynamic load have been studied for many years by several authors (Huth, 1955; Baker, 1961; Baker *et al.*, 1966; Mckinney, 1971; Rose *et al.*, 1973; Pan and Chow, 1976; Pao and Ceranoglu, 1978; Pao, 1983). However, if the sphere is not isotropic, the cases so far studied are much fewer in number because the solving process is more complex.

In this paper, the elastodynamic equation for an anisotropic hollow sphere is derived and a simple solution is presented. The anisotropic elastodynamic equation is decomposed into a quasi-static equation with inhomogeneous boundary conditions and a dynamic equation with homogeneous boundary conditions. By using the method described in Lekhnitskii (1981), we can solve the anisotropic quasi-static equation. The solution to the anisotropic dynamic equation can be obtained by means of finite Hankel transform (Cinelli, 1965). Thus, the solution for an anisotropic elastodynamical sphere is rigorously derived.

Finally, we calculate some practical examples in which an anisotropic hollow sphere is subjected to a uniform sudden load and an exponential decaying shock. The histories and distributions of the dynamic stresses are given and the features of the solution which are related to the propagation of the spherical wave and radial vibrations of an anisotropic sphere are discussed.

2. ANISOTROPIC ELASTODYNAMIC EQUATION AND METHOD OF SOLUTION

We start this paper by considering an anisotropic hollow sphere (spherical vessel) acted on by a dynamic internal and external pressure $\psi_1(t)$, $\psi_2(t)$ distributed uniformly over the surface. The investigation is most conveniently carried out by using a spherical coordinate system r, θ, φ with the origin at the centre of the sphere. The material of a given sphere is assumed to possess transverse isotropy about any radius vector drawn from the common centre of the sphere to a given point. It is obvious that in the case of the elastic properties indicated, the distribution of stress and strain depends only on the radial variable r, and all points are displaced only in radial directions during deformation. Denoting the single (radial) component of displacement by $U(r)$, we have the strain components referred to spherical coordinates

$$\varepsilon_r = \frac{\partial U}{\partial r}, \quad \varepsilon_\theta = \varepsilon_\varphi = \frac{U}{r}$$

$$\gamma_{\theta\varphi} = \gamma_{r\varphi} = \gamma_{r\theta} = 0. \tag{1}$$

Since all shear deformations are zero, it follows that the shear stresses are zero. If the engineering constants are introduced, the generalized Hooke's law takes the form

$$\sigma_r = A_{11}\varepsilon_r + 2A_{12}\varepsilon_\theta$$
$$\sigma_\theta = A_{12}\varepsilon_r + (A_{22}+A_{23})\varepsilon_\theta, \tag{2}$$

where elastic moduli A_{ij} is related to E and v as follows:

$$A_{11} = \frac{E_r(1-v_\theta)}{m}, \quad A_{12} = \frac{E_\theta v_r}{m}$$

$$A_{22} = \frac{E_\theta}{(1+v_\theta)m}\left(1 - v_r^2\frac{E_\theta}{E_r}\right)$$

$$A_{23} = \frac{E_\theta}{(1+v_\theta)m}\left(v_\theta + v_r^2\frac{E_\theta}{E_r}\right)$$

$$m = 1 - v_\theta - 2v_r^2\frac{E_\theta}{E_r}. \tag{3}$$

In the above formulas, E_r and E_θ are Young's moduli for tension along a radius vector r and in a direction perpendicular to it, v_r is Poisson's ratio characterizing transverse contraction in a direction perpendicular to r when tension is applied in the r direction and v_θ is Poisson's ratio characterizing contraction in a plane normal to a radius vector for tension in the same plane. Only one of the equilibrium equations for a continuous body referred to a spherical coordinate system remains

$$\frac{\partial \sigma_r}{\partial r} + \frac{2(\sigma_r - \sigma_\theta)}{r} = \rho\frac{\partial^2 U}{\partial t^2} \tag{4a}$$

with the following boundary conditions:

$$\sigma_r(a,t) = \psi_1(t) \tag{4b}$$

$$\sigma_r(b,t) = \psi_2(t), \tag{4c}$$

where a and b are the inner and outer radii, respectively, and ρ is the mass density. From (1), (2) and (4), we obtain an anisotropic elastodynamic equation and boundary conditions for the displacement

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} - 2\frac{A_{22}+A_{23}-A_{13}}{A_{11}}\frac{U}{r^2} = \frac{1}{V^2}\frac{\partial^2 U}{\partial t^2} \tag{5a}$$

$$\sigma_r(a,t) = A_{11}\frac{\partial U(a,t)}{\partial r} + A_{12}\frac{2U(a,t)}{a} = \psi_1(t) \tag{5b}$$

$$\sigma_r(b,t) = A_{11}\frac{\partial U(b,t)}{\partial r} + A_{12}\frac{2U(b,t)}{b} = \psi_2(t) \tag{5c}$$

where $V = \sqrt{(A_{11}/\rho)}$ is the spherical wave speed. The initial conditions are expressed as

$$U(r,0) = 0, \quad \dot{U}(r,0) = 0. \tag{5d}$$

Let us suppose that

$$U(r,t) = U_s(r,t) + U_d(r,t) \tag{6}$$

where $U_s(r,t)$ is a quasi-static solution of the basic eqn (5a), which satisfies the following quasi-static equation and inhomogeneous boundary conditions

$$\frac{\partial^2 U_s}{\partial r^2} + \frac{2}{r}\frac{\partial U_s}{\partial r} - 2\frac{A_{22}+A_{23}-A_{13}}{A_{11}}\frac{U_s}{r^2} = 0 \tag{7a}$$

$$A_{11}\frac{\partial U_s(a,t)}{\partial r} + 2A_{12}\frac{U_s(a,t)}{a} = \psi_1(t) \tag{7b}$$

$$A_{11}\frac{\partial U_s(b,t)}{\partial r} + 2A_{12}\frac{U_s(b,t)}{b} = \psi_2(t). \tag{7c}$$

The general integral of eqn (7) is of the form

$$U_s(r,t) = \Phi_1(r)\psi_1(t) + \Phi_2(r)\psi_2(t). \tag{8a}$$

In the above formula, we have

$$\Phi_1(r) = C_1 r^{n-0.5} + C_2 r^{-n-0.5} \tag{8b}$$

$$\Phi_2(r) = C_3 r^{n-0.5} + C_4 r^{-n-0.5}, \tag{8c}$$

where

$$C_1 = \frac{-C^{1.5+n}b^{1.5-n}}{(1-C^{2n})[2A_{12}+A_{11}(n-0.5)]} \tag{9a}$$

$$C_2 = \frac{C^{1.5+n}b^{1.5+n}}{(1-C^{2n})[2A_{12}-A_{11}(n+0.5)]} \tag{9b}$$

$$C_3 = -C^{-n-1.5}C_1 \tag{9c}$$

$$C_4 = -C^{n-1.5}C_2 \tag{9d}$$

$$n = \sqrt{\frac{1}{4} + 2\frac{A_{22}+A_{23}-A_{12}}{A_{11}}} \tag{9e}$$

$$C = a/b. \tag{9f}$$

Substituting (6) into (5) and utilizing (7) provides an inhomogeneous dynamic equation with homogeneous boundary conditions for $U_d(r,t)$

$$\frac{\partial^2 U_d}{\partial r^2} + \frac{2}{r}\frac{\partial U_d}{\partial r} - 2\frac{A_{22}+A_{23}-A_{13}}{A_{11}}\frac{U_d}{r^2} = \frac{1}{V^2}\left[\frac{\partial^2 U_d}{\partial t^2} - \frac{\partial^2 U_s}{\partial t^2}\right] \tag{10a}$$

$$A_{11}\frac{\partial U_d(a,t)}{\partial r} + 2A_{12}\frac{U_d(a,t)}{a} = 0 \tag{10b}$$

$$A_{11}\frac{\partial U_d(b,t)}{\partial r} + 2A_{12}\frac{U_d(b,t)}{b} = 0 \tag{10c}$$

$$U_d(r,0) = 0, \quad \dot{U}_d(r,0) = 0. \tag{10d,e}$$

Suppose that

$$U_d(r,t) = r^{-1/2}f(r,t). \tag{11}$$

Substituting eqn (11) into (10), yields

$$\frac{\partial^2 f(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial f(r,t)}{\partial r}-\frac{k^2}{r^2}f(r,t) = \frac{1}{V^2}\left[\frac{\partial^2 f(r,t)}{\partial t^2}+\frac{\partial^2 U_{s1}}{\partial t^2}\right] \tag{12a}$$

$$\frac{\partial f(a,t)}{\partial r}+\frac{4A_{12}-A_{11}}{2A_{11}a}f(a,t) = 0 \tag{12b}$$

$$\frac{\partial f(b,t)}{\partial r}+\frac{4A_{12}-A_{11}}{2A_{11}b}f(b,t) = 0 \tag{12c}$$

$$f(r,0) = 0, \quad \dot{f}(r,0) = 0, \tag{12d,e}$$

where

$$k^2 = \frac{3(A_{22}+A_{23}-A_{13})}{2A_{11}} \tag{13a}$$

$$U_{s1}(r,t) = r^{1/2}U_s(r,t) \tag{13b}$$

and $U_s(r,t)$ is the known solution as shown in (8).

The homogeneous equation (let $U_s = 0$) of eqn (12a) with homogeneous boundary (12b, c) is solved by assuming

$$f_{d0}(r,t) = g(r)\exp(\mathrm{j}wt). \tag{14}$$

From (12) and (14), we have the following eigen-equation

$$Y_aJ_b - Y_bJ_a = 0, \tag{15a}$$

where

$$Y_a = \xi_t Y_k'(\xi_t a)+h_a Y_k(\xi_t a), \quad J_a = \xi_t J_k'(\xi_t a)+h_a J_k(\xi_t a)$$

$$Y_b = \xi_t Y_k'(\xi_t b)+h_b Y_k(\xi_t b), \quad J_b = \xi_t J_k'(\xi_t b)+h_b J_k(\xi_t b)$$

$$h_a = \frac{4A_{12}-A_{11}}{2aA_{11}}, \quad h_b = \frac{4A_{12}-A_{11}}{2bA_{11}}. \tag{15b–g}$$

$J_k(\xi_i r)$ and $Y_k(\xi_i r)$ are the first and second kind of the kth-order Bessel function, respectively. In these expressions, ξ_i $(i = 1,2,\ldots,n)$ express a series of positive roots for natural eigen-equation (15a). The natural frequencies are

$$\omega_t = V\xi_t. \tag{16}$$

From Cinelli (1965), defining $\bar{f}(\xi_t,t)$ as the finite Hankel transform of $f(r,t)$ yields

$$\bar{f}(\xi_t,t) = H[f(r,t)] = \int_a^b rf(r,t)G(\xi_t r)\,\mathrm{d}r. \tag{17}$$

Then, by making use of the inverse of the transform, we have

$$f(r,t) = \sum_t F(\xi_t)\bar{f}(\xi_t,t)G(\xi_t r), \tag{18a}$$

where

$$F(\xi_t) = \frac{1}{\int_a^b r[G(\xi_t r)]^2 \, dr} \tag{18b}$$

$$G(\xi_t r) = J_k(\xi_t r) Y_a - Y_k(\xi_t r) J_a. \tag{18c}$$

Applying the finite Hankel transform (17) to (12a), we have

$$\frac{2}{\pi}\frac{J_a}{J_b}[f(b) + h_b f(b)] - \frac{2}{\pi}[f(a) + h_a f(a)] - \xi_t \bar{f} = \frac{1}{V^2}\left[\frac{\partial^2 \bar{f}}{\partial t^2} + \frac{\partial^2 \bar{U}_s}{\partial t^2}\right], \tag{19}$$

where

$$\bar{U}_s = H[\bar{U}_{s1}].$$

The first two terms on the left-hand side of (19) should be equal to zero in view of the homogeneous boundary conditions (12b, c). Thus, (19) is reduced to

$$-\xi_t^2 \bar{f}(\xi_t, t) = \frac{1}{V^2}\left[\frac{d^2 \bar{f}}{dt^2} + \frac{d^2 \bar{U}_s}{dt^2}\right] \tag{20}$$

Applying the Laplace transform to eqn (20), we obtain

$$\bar{f}^* = -\bar{U}_s^* + \frac{\xi_t^2 V^2}{\xi_t^2 V^2 + p^2} \bar{U}_s^*, \tag{21}$$

where p is the parameter of the Laplace transform. The inverse Laplace transform for (21) gives

$$\bar{f}(\xi_t, t) = \bar{\Phi}_1(\xi_t) I_{1t}(\xi_t, t) + \bar{\Phi}_2(\xi_t) I_{2t}(\xi_t, t) \tag{22}$$

$$I_t(\xi_t, t) = -\psi_j(t) + \xi_t V \int_0^t \psi_j(\tau) \sin[\xi_t V(t-\tau)] \, d\tau \tag{23a}$$

$$\bar{\Phi}_j(\xi_t) = H[\Phi_j(r)], \quad j = 1, 2. \tag{23b}$$

From formulas (18), the solution of $f(r, t)$ is obtained as follows:

$$f(r, t) = \sum_t F(\xi_t) G(\xi_t r) \bar{f}(\xi_t, t). \tag{24}$$

Substituting (24) into (11), we get the dynamic solution for inhomogeneous dynamic equation with homogeneous boundary conditions

$$U_d(r, t) = \sum_t r^{-1/2} F(\xi_t) G(\xi_t r) \bar{f}(\xi_t, t). \tag{25}$$

Thus, from (6), (8) and (25), the solution of the anisotropic elastodynamic equation (5) is written as

$$U(r, t) = \Phi_1(r)\psi_1(t) + \Phi_2(r)\psi_2(t) + \sum_t r^{-1/2} F(\xi_t) G(\xi_t r) \bar{f}(\xi_t, t). \tag{26}$$

The stress field is easily found from formulas (26), (2) and (1).

3. EXAMPLES AND DISCUSSIONS

As examples, results are carried out in the case of an anisotropic hollow sphere with $b/a = 2$. The material of a given body is assumed to possess transverse isotropy along any radius vector drawn from the common centre of the spheres to a given point. The material properties are specified as $E_r = 200$ GPa, $E_\theta/E_r = 25/9$, $\nu_r = \nu_\theta = 0.25$. The mass density is $\rho = 0.0096\ \text{N/cm}^3$.

Note that the solution (26) comprises two parts. The first part is a quasi-static part and the second part is the dynamic one which is associated to the radial vibration and propagation of spherical waves in an anisotropic sphere. The positive roots of eqn (15) which give the characteristic values ξ_i and the natural frequencies ω_i are calculated. The first ten values of ω_i and the increments $\Delta\omega_i$ for an anisotropic hollow sphere and an isotropic hollow sphere are listed in Table 1. The natural frequencies ω_i of the material properties with $E_\theta = 25E_r/9$ are larger than that of the material properties with $E_\theta = E_r$. No matter what the material properties are the differences of $\Delta\omega_i$ become smaller and smaller as the mode number i gets larger and larger. It means that the natural frequencies ω_i increase with approximately a constant value versus the mode number i, when such a parameter is sufficiently high. From Table 2, we see that the natural frequencies ω_i also vary with the ratio b/a no matter what the material properties are. The first-order frequency ω_1 decreases when b/a increases.

We assume that only the internal boundary of an anisotropic sphere is subjected to a dynamic stress $\psi_1(t)$. In this case, we have

$$\psi_1(t) = -\sigma_0 \exp(-\alpha t), \quad t \geqslant 0^+. \tag{27}$$

In such an expression when α equals zero, the dynamic load $\psi_1(t)$ is a sudden uniform pressure form. When α is not equal to zero, the dynamic load $\psi_1(t)$ becomes an exponential decaying shock pressure. In this case, we suppose $\alpha = 500$. In all results, the stresses are normalized by the amplitude of applied pressure σ_0, then $\bar{\sigma}_t = \sigma_t/\sigma_0$, the wall thickness ratio is $b/a = 2$, the time period $T = tV/(b-a)$, $R = (r-a)/(b-a)$, and Δ expresses corresponding static stress.

Figure 1 shows the histories and distribution of the dynamic stresses for an isotropic sphere. In this case, $E_r = 200$ GPa, $E_\theta = E_r$, $\alpha = 0$. Figure 2 shows the histories and distribution of the dynamic stresses for an anisotropic sphere. In this case, $E_r = 200$ GPa, $E_\theta = 25E_r/9$, $\alpha = 0$. Comparing Fig. 1 with Fig. 2(a, b), we see that the radial stress response of the isotropic sphere in Fig. 1(a) is a close approximation to that of anisotropic sphere in Fig. 2(a) because the radial material properties of two spheres are the same. However, the tangential stress amplitude of the anisotropic sphere in Fig. 2(b) is larger than that of the isotropic sphere in Fig. 1(b), because the tangential properties of the two spheres are very different.

The histories and distributions of the dynamic stress for anisotropic spherical shells subjected to an exponential decaying shock pressure where $\alpha = 500$ are shown in Fig. 3.

Table 1. The first ten values of ω_i and $\Delta\omega_i$ for $b/a = 2$

	i	1	2	3	4	5	6	7	8	9	10
$(E_\theta/E_r) = 1$	$\omega_i(1/s)$	5091	16,813	31,954	47,480	63,098	78,752	94,425	110,108	125,797	141,490
	$\Delta\omega_i(1/s)$	11,722	15,141	15,526	15,618	15,654	15,673	15,683	15,689	15,693	
$(E_\theta/E_r) = 25/9$	$\omega_i(1/s)$	9724	21,625	40,164	59,374	78,756	98,206	117,693	137,198	156,717	176,244
	$\Delta\omega_i(1/s)$	11,901	18,539	19,210	19,382	19,450	19,487	19,505	19,519	19,527	

Table 2. The lowest ω_i for different ratios of b/a

	b/a	2	4	6	8	10	20	40	60	80	100
ω_i	$(E_\theta/E_r) = 1$	5091	3043	2100	1590	1277	640	320	213	160	128
$(1/s)$	$(E_\theta/E_r) = 25/9$	9724	6348	4329	3255	2571	1216	572	412	288	212

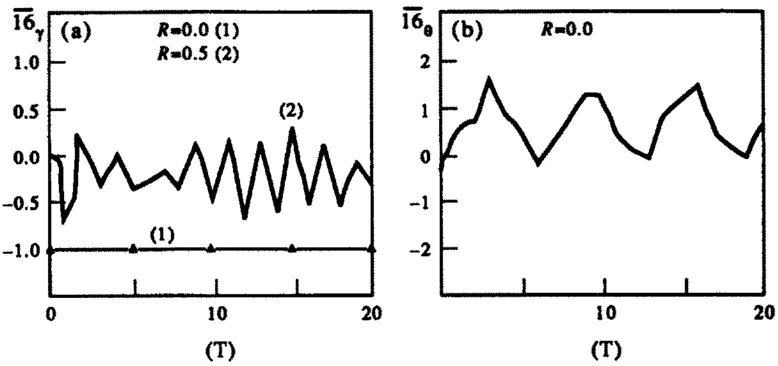

Fig. 1. The isotropic hollow sphere, $E_r = E_\theta = 200$ GPa, $\alpha = 0$, $\bar{\sigma}_i = \sigma_i/\sigma_0$, $T = tV/(b-a)$, $R = (r-a)/(b-a)$.

Because of the small wall thickness, the effects of the wave reflected between the inner and outer walls on dynamic stresses must be considered. Except for the radial stress at the inner boundary, where $\bar{\sigma}_r = -\exp(-\alpha t)$ as shown in Fig. 3(a), the stresses at the other points oscillate dramatically as shown in Figs 2 and 3.

Because the oscillations are accompanied with stress waves propagating between the inner and outer boundaries, where the reflected waves are produced successively upon the arrival of the incident waves, the numerical results concerning the distribution of the stress fields vs time presented in all the figures possess sharp profiles similar to those given by Hata (1991).

The histories of the radial and tangential stresses at $r = a$ are shown in the (a) and (b) parts of the figures, respectively. It should be mentioned that the maximum amplitude of the tangential stress at $r = a$ is much larger than that of the radial stress at $r = a$; this is

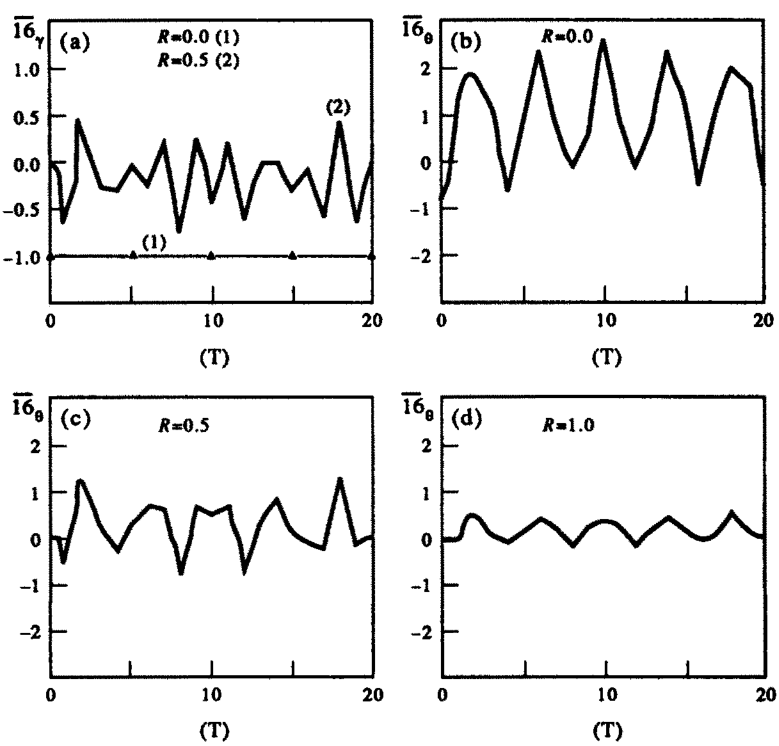

Fig. 2. The anisotropic hollow sphere, $E_r = 200$ GPa, $E_\theta = 25E_r/9$, $\alpha = 0$, $\bar{\sigma}_i = \sigma_i/\sigma_0$, $T = tV/(b-a)$, $R = (r-a)/(b-a)$.

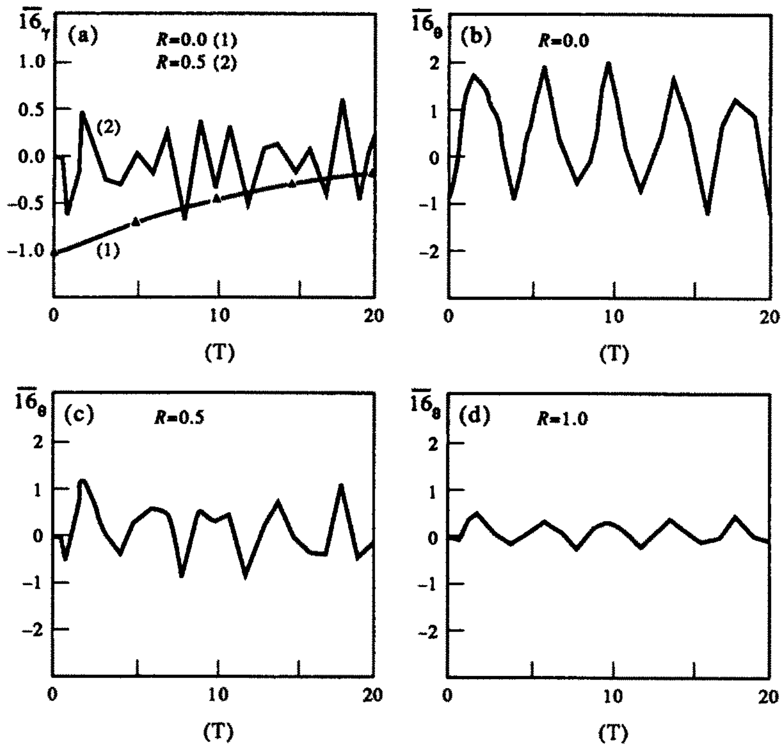

Fig. 3. The anisotropic hollow sphere, $E_r = 200$ GPa, $E_\theta = 25E_r/9$, $\alpha = 500$, $\bar{\sigma}_i = \sigma_i/\sigma_0$, $T = tV/(b-a)$, $R = (r-a)/(b-a)$.

because the tangential stiffness is much larger than the radial stiffness of a spherical shell. Because of the wave property of strong discontinuity, the sign of the tangential stress at the wavefront is reversed as compared to that of the static stress from Fig. 2(b) and Fig. 3(b). This is similar to the result of an isotropic cylindrical cavity in infinite elastic medium under impact pressure given by Selberg (1952).

From eqn (26) we can see that the solution is composed of an anisotropic static solution and a dynamic solution with homogeneous boundary conditions. The effects of the reflected wave mean that the histories of stresses oscillate dramatically around the static stress. The oscillating amplitude of the stress mainly depends on the loading rate, but not the loading amplitude. On the other hand, from $\alpha = 0$ to 500, the source spectrum is effectively changed only as time t increases dramatically. At $t = 0$, the loading amplitude for $\alpha = 0$ is the same as that for $\alpha = 500$. Considering the above reasons the results presented in Figs 2 and 3 have only a small difference when the time t is less, and the loading rate is the same.

Finally, we conclude that the present solution is valid theoretically. The results obtained can be used as a basis for the assessment of various approximate theories.

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